

# The kernel of the adjoint representation of a $p$ -adic Lie group need not have an abelian open normal subgroup

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## Abstract

Let  $G$  be a  $p$ -adic Lie group with Lie algebra  $\mathfrak{g}$  and  $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$  be the adjoint representation. It was claimed in the literature that the kernel  $K := \ker(\text{Ad})$  always has an abelian open normal subgroup. We show by means of a counterexample that this assertion is false; it can even happen that  $K = G$  but  $G$  has no abelian subnormal subgroup except for the trivial group. The arguments are based on auxiliary results on subgroups of free products with central amalgamation.

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## Introduction and statement of results

The context of these investigations are recent studies of normal series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$$

of closed subgroups in a  $p$ -adic Lie group  $G$ . Although composition series need not exist (as is evident from  $\mathbb{Z}_p \triangleright p\mathbb{Z}_p \triangleright p^2\mathbb{Z}_p \triangleright \cdots$ ), by reasons of dimension one can always find a series all of whose subquotients  $Q := G_{j-1}/G_j$  are *nearly simple* in the sense that each closed subnormal subgroup  $S \subseteq Q$  is open or discrete [4]. It turns out that a quite limited list of subquotients suffices to build up  $G$  (*loc. cit.*). The kernel of the adjoint representation is an important source of normal subgroups, and its properties have an impact on the list of subquotients needed to build up general  $p$ -adic Lie groups (see the proposition below). Some statements from the literature are relevant in this connection. In [3, Exercise 9.12], the following assertion is made:

**Assertion 1.** *Let  $G$  be a  $p$ -adic Lie group. Then  $G$  has closed normal subgroups  $Z \subseteq K$  such that  $Z$  is abelian,  $K/Z$  is discrete, and  $G/K$  is isomorphic to a subgroup of  $\mathrm{GL}_d(\mathbb{Q}_p)$  where  $d = \dim(G)$ .*

The hint given in the exercise amounts to the following assertion:

**Assertion 2.** *If  $G$  is a  $p$ -adic Lie group, with Lie algebra  $\mathfrak{g}$  and adjoint representation  $\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})$ , then the kernel  $\ker(\mathrm{Ad})$  has an abelian open subgroup which is normal in  $G$ .*

In fact, the hint to solve the exercise is to take  $K := \ker(\mathrm{Ad})$  and define  $G^+$  as the subgroup generated by all compact open subgroups of  $G$  which are uniformly powerful. This is an open normal subgroup of  $G$  (and so  $G/G^+$  and  $K/(K \cap G^+)$  are discrete). The reader is supposed to show that  $K$  is the centralizer of  $G^+$  in  $G$  (which is false), which would entail that  $K$  and  $Z := K \cap G^+$  can be used to obtain Assertion 1.

The goal of this note is to show that Assertions 1 and 2 are false, and to record additional pathologies that may occur. However, a weakened version of Assertion 1 cannot be ruled out so far:

**Problem 1.** Does every  $p$ -adic Lie group  $G$  have closed subgroups  $Z \triangleleft K \triangleleft G$  such that  $Z$  is abelian,  $K/Z$  is discrete and  $K = \ker \beta$  for a continuous homomorphism  $\beta: G \rightarrow \mathrm{GL}_d(\mathbb{Q}_p)$  for some  $d \in \mathbb{N}$ ?

Our counterexample to Assertions 1 and 2 reads as follows:

**Theorem.** *The ascending union*

$$G := \bigcup_{n \in \mathbb{N}_0} (\mathbb{Q}_p *_{\mathbb{Z}_p} \mathbb{Q}_p *_{p\mathbb{Z}_p} \cdots *_{p^n \mathbb{Z}_p} \mathbb{Q}_p)$$

*of iterated amalgamated products can be made a 1-dimensional  $p$ -adic Lie group with the following properties:*

- (a)  $\mathrm{Ad}(g) = \mathrm{id}$  for all  $g \in G$ , i.e.,  $\ker(\mathrm{Ad}) = G$ ;
- (b)  $G$  does not have an abelian non-trivial subnormal subgroup;
- (c)  $G$  has a discrete normal subgroup  $K$  such that  $G/K \cong \mathbb{Q}_p$ , whence  $K = \ker(\psi)$  for some analytic homomorphism  $\psi: G \rightarrow \mathrm{GL}_2(\mathbb{Q}_p)$ ; and:
- (d) The commutator group  $G'$  is discrete.

**Remarks.**

- (i) By (a) and (b),  $\ker(\text{Ad}) = G$  does not have an abelian open normal subgroup, showing that Assertion 2 is false.
- (ii) Our group  $G$  also shows that Assertion 1 is false.<sup>1</sup>
- (iii) Our group  $G$  does not provide a negative answer to Problem 1, as we can choose  $K$  as in (c) and let  $Z$  be the trivial group.
- (iv) As a consequence of (b), neither  $G$ , nor any non-trivial subnormal subgroup of  $G$ , is soluble.

Following [4], a nearly simple  $p$ -adic Lie group  $H$  is called *extraordinary* if  $H = \ker(\text{Ad})$  holds and  $S'$  is non-discrete for each open subnormal subgroup  $S$  of  $H$ . This implies that the radical  $R(S)$  is discrete for each open subnormal subgroup  $S$  of  $H$  [4, Remark 3.3].<sup>2</sup> It is not known whether extraordinary  $p$ -adic Lie groups exist. In Section 4, we prove:

**Proposition.** *If Problem 1 has a positive answer, then extraordinary  $p$ -adic Lie groups cannot exist.*

Being 1-dimensional, the Lie group  $G$  from the above theorem is nearly simple. Also,  $G = \ker \text{Ad}$  and  $R(S) = \mathbf{1}$  (which is discrete), for each subnormal subgroup  $S$  of  $G$ . Thus  $G$  shares many properties of extraordinary groups. Yet,  $G$  is not extraordinary, as  $G'$  is discrete (by (d) in the theorem).

## 1 Preliminaries

We write  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . If  $G$  is a group, then  $Z(G)$  denotes its centre. As usual, we write  $G' \subseteq G$  for the commutator group of  $G$ , generated by the commutators  $ghg^{-1}h^{-1}$  for  $g, h \in G$  and define the

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<sup>1</sup>In fact,  $d = \dim(G) = 1$  here. Suppose that  $K := \ker \beta$  had an abelian open normal subgroup  $Z$  for some homomorphism  $\beta: G \rightarrow \text{GL}_1(\mathbb{Q}_p)$ . Since  $G$  has no non-trivial abelian subnormal subgroups, we must have  $Z = \mathbf{1}$ , whence  $K$  would be discrete and thus  $K \neq G$ . Since  $\mathbb{Z}_p^\times$  (being pro-finite) and  $\mathbb{Z}$  do not contain divisible elements apart from the neutral element but  $\mathbb{Q}_p$  is divisible, every group homomorphism  $\mathbb{Q}_p \rightarrow \text{GL}_1(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$  is trivial. Since  $G$  is generated by copies of  $\mathbb{Q}_p$ , we deduce that every group homomorphism  $G \rightarrow \text{GL}_1(\mathbb{Q}_p)$  is trivial. Thus  $K = G$  is non-discrete, contradiction.

<sup>2</sup>The notion of radical used here is recalled in Section 1.

derived series of  $G$  via  $G^{(0)} := G$ ,  $G^{(n+1)} := (G^{(n)})'$  for  $n \in \mathbb{N}_0$ . We write  $\mathbf{1}$  for the trivial group. As usual,  $G$  is called *soluble* if  $G^{(n)} = \mathbf{1}$  for some  $n \in \mathbb{N}_0$ . The *radical*  $R(G)$  of  $G$  is defined as the union of all soluble normal subgroups of  $G$ . Then  $R(G)$  is a normal (and, in fact, characteristic) subgroup of  $G$ .

Let  $A_1$  and  $A_2$  be groups having isomorphic subgroups  $B_1 \subseteq A_1$  and  $B_2 \subseteq A_2$ , respectively. Let  $\theta: B_1 \rightarrow B_2$  be an isomorphism. Then the amalgam  $A_1 *_\theta A_2$  can be defined (see [5]). We shall always assume that  $B := B_1 = B_2$  and  $\theta = \text{id}_B$  is the identity map. As usual, we abbreviate

$$A_1 *_B A_2 := A_1 *_{\text{id}_B} A_2$$

and call  $A_1 *_B A_2$  the *free product of  $A_1$  and  $A_2$  with amalgamated subgroup  $B$*  (or *amalgamated product*, for short). We are only interested in central amalgamations, i.e., we shall always assume that  $B \subseteq Z(A_1)$  and  $B \subseteq Z(A_2)$ . For basic properties of amalgamated products, the reader is referred to [5]. In particular, we essentially use [5, Theorem 11.66] on the normal form for elements of  $A_1 *_B A_2$ . See [1] and [6] for basic theory and notation concerning  $p$ -adic Lie groups. We shall write  $L(G)$  for the Lie algebra of a  $p$ -adic Lie group  $G$  and  $\text{rad}(\mathfrak{g})$  for the radical of a finite-dimensional  $p$ -adic Lie algebra  $\mathfrak{g}$  (the largest soluble ideal). A  $p$ -adic Lie group  $G$  is called *linear* if it admits an injective continuous homomorphism  $G \rightarrow \text{GL}_d(\mathbb{Q}_p)$  for some  $d \in \mathbb{N}$ . If  $G$  is linear, then  $L(R(G)) = \text{rad}(L(G))$  ([4, Lemma 1.3]; cf. [2, Lemma 6.4]).

## 2 Auxiliary results on amalgamated products

**Lemma 2.1** *Let  $A$  and  $C$  be groups and  $B$  be a subgroup of both  $Z(A)$  and  $Z(C)$ . If  $h \in A$  such that  $h \notin B$  and  $g \in A *_B C$  such that  $g \notin A$ , then  $ghg^{-1} \notin A$  and  $ghg^{-1}h^{-1} \notin A$ .*

**Proof.** We may assume that the sets  $A \setminus B$  and  $C \setminus B$  are disjoint. Let  $R \subseteq A$  be a transversal for  $A/B$  and  $S \subseteq C$  be a transversal for  $C/B$ . Then  $h = rb$  for unique  $r \in R$  and  $b \in B$ . Moreover,

$$g = x_1 \cdots x_n b'$$

for some  $n \in \mathbb{N}$ ,  $b' \in B$  and elements  $x_1, \dots, x_n \in R \cup S$  such that  $x_i \in R \Rightarrow x_{i+1} \in S$  and  $x_i \in S \Rightarrow x_{i+1} \in R$  for all  $i \in \{1, \dots, n-1\}$ . Since  $g \notin A$ , we have  $x_i \in S$  for some  $i \in \{1, \dots, n\}$ . We let  $j$  be the minimum of all such  $i$ ,

and  $k$  be their maximum. Then  $j \in \{1, 2\}$ ,  $j \leq k$ ,  $k \in \{n-1, n\}$  and both  $x_j, x_k \in S$ .

We show that  $ghg^{-1} \notin A$ . Then also  $ghg^{-1}h^{-1} \notin A$  (as  $h \in A$ ).

If  $ghg^{-1} \notin A$  was false, we would have  $gh = ag$  for some  $a \in A$ . Note that

$$gh = x_1 \dots x_n r b b'.$$

If  $k = n$ , then

$$gh = x_1 \dots x_n r b_1 \tag{1}$$

is in normal form, with  $b_1 := b b' \in B$ .

If  $k = n-1$  and  $x_n r \in B$ , then

$$gh = x_1 \dots x_{n-1} b_2 \tag{2}$$

is in normal form, with  $b_2 := x_n r b b' \in B$ .

If  $k = n-1$  and  $x_n r \notin B$ , then  $x_n r b b' = r' b_3$  for some  $r' \in R$  and  $b_3 \in B$ , and thus

$$gh = x_1 \dots x_{n-1} r' b_3 \tag{3}$$

is in normal form.

Suppose that  $a \in B$ . We then deduce from

$$gh = ag = ga$$

(where we used that  $B \subseteq Z(A *_B C)$ ) that  $h = a \in B$ . But  $h \notin B$ , contradiction. We therefore must have  $a \in A \setminus B$  and thus  $a = r'' b''$  with  $r'' \in R$  and  $b'' \in B$ .

If  $j = 1$ , this implies that

$$ag = r'' x_1 \dots x_n b_4 \tag{4}$$

is in normal form, with  $b_4 := b' b''$ . Then  $gh = ag$  is false (contradiction) because the normal form (4) differs from the ones in (1), (2) and (3) (each starting with a representative  $x_1 \in S$ , as we assume  $j = 1$ ).

If  $j = 2$  and  $r'' x_1 \in B$ , then

$$ag = x_2 \dots x_n b_5 \tag{5}$$

is in normal form, with  $b_5 := r''x_1b'b'' \in B$ . Again we obtain  $gh \neq ag$ , as the normal form in (5) differs from the ones in (1) and (3) (which involve  $n+1$  and  $n$  representatives, respectively); it also differs from the normal form in (2) as the latter ends with a representative in  $S$ , while the normal form in (5) ends with a representative in  $R$  in the situation of (2).

If  $j = 2$  and  $r''x_1 \notin B$ , then

$$r''x_1b'b'' = r_1b_6 \quad (6)$$

with  $r_1 \in R$  and  $b_6 \in B$ , and thus

$$ag = r_1x_2 \cdots x_nb_6 \quad (7)$$

is in normal form. Again  $gh \neq ag$ , as the normal form in (7) differs from that in (1) (which involves  $n+1$  representatives) and the one in (2) (which only involves  $n-1$  representatives). It also differs from the normal form in (3). In fact, if we had  $gh = ag$  in the situation of (3) and (7), then

$$x_1 \cdots x_{n-1}r'b_3 = r_1x_2 \cdots x_nb_6$$

would imply that  $x_1 = r_1$ ,  $r' = x_n$  and  $b_3 = b_6$ . Thus  $r''x_1b'b'' = r_1b_6 = x_1b_6$  (using (6)), whence  $b'b''r''x_1 = b_6x_1$  and thus  $b'b''r'' = b_6$ . Hence  $r'' = (b'b'')^{-1}b_6 \in B$ , contrary to  $r'' \in R \subseteq A \setminus B$ . So, in all cases,  $ghg^{-1} \notin A$ .  $\square$

**2.2** Throughout the rest of this section, we consider the following situation:

- (a)  $B_0 \supseteq B_1 \supseteq \cdots$  is a descending sequence of groups with  $\bigcap_{n \in \mathbb{N}_0} B_n = \mathbf{1}$ ;
- (b)  $(H_n)_{n \in \mathbb{N}_0}$  is a sequence of groups such that, for all  $n \in \mathbb{N}_0$ , both  $Z(H_n)$  and  $Z(H_{n+1})$  contain  $B_n$  as a subgroup; and
- (c)  $B_n \neq H_n$  and  $B_n \neq H_{n+1}$ , for each  $n \in \mathbb{N}_0$ .

We define  $G_0 := H_0$  and  $G_n := G_{n-1} *_{B_{n-1}} H_n$  for  $n \in \mathbb{N}$ . Then  $G_0 \subseteq G_1 \subseteq \cdots$  and we define the group  $G$  as the union (direct limit)

$$G := \bigcup_{n \in \mathbb{N}_0} G_n = \bigcup_{n \in \mathbb{N}_0} H_0 *_{B_0} H_1 *_{B_1} \cdots *_{B_{n-1}} H_n.$$

**Definition 2.3** A subgroup  $H \subseteq G$  is called *spread out* if

$$(\forall n \in \mathbb{N}) \quad H \not\subseteq G_n.$$

For example,  $G$  is spread out.

**Lemma 2.4** *If a subgroup  $H \subseteq G$  is spread out, then every non-trivial normal subgroup  $N$  of  $H$  is spread out.*

**Proof.** Let  $k \in \mathbb{N}_0$ . We show that  $N$  is not a subset of  $G_k$ . Let  $e \neq h \in N$  and  $m \in \mathbb{N}_0$  such that  $h \in G_m$ . After increasing  $m$  if necessary, we may assume that  $m \geq k$ . Since  $\bigcap_{n \in \mathbb{N}_0} B_n = \mathbf{1}$ , after increasing  $m$  further we may assume that  $h \notin B_m$ . Since  $H$  is spread out, there exists  $g \in H$  such that  $g \notin G_m$ . There exists  $\ell > m$  such that  $g \in G_\ell$ . We choose  $\ell$  minimal. Then  $g \notin G_{\ell-1}$ . After increasing  $m$ , we may assume that  $m = \ell - 1$ . Thus  $g \in G_{m+1}$  but  $g \notin G_m$ . Now  $ghg^{-1} \in N$  as  $N$  is a normal subgroup of  $H$ . Moreover, applying Lemma 2.1 to the subgroup  $G_{m+1} = G_m *_{B_m} H_{m+1}$  of  $G$ , we see that  $ghg^{-1} \notin G_m$ . Since  $m \geq k$ , we deduce that  $ghg^{-1} \notin G_k$ . Thus  $N$  is spread out, which completes the proof.  $\square$

**Remark 2.5** For later use, note that also  $ghg^{-1}h^{-1} \notin G_m$  in the preceding proof (since  $ghg^{-1} \notin G_m$  and  $h \in G_m$ ). Thus  $ghg^{-1}h^{-1} \notin G_k$  (as  $m \geq k$ ).

A trivial induction based on Lemma 2.4 shows:

**Lemma 2.6** *If a subgroup  $H$  of  $G$  is spread out, then also every non-trivial subnormal subgroup  $S$  of  $H$  is spread out. In particular, every non-trivial subnormal subgroup  $S$  of  $G$  is spread out.*  $\square$

**Lemma 2.7** *If a subgroup  $H \subseteq G$  is spread out, then  $H' \neq \mathbf{1}$  and thus  $H$  is not abelian.*

**Proof.** Let  $k \in \mathbb{N}_0$ . Taking  $N = H$  in the proof of Lemma 2.4, we obtain  $g, h \in H$  such that  $ghg^{-1}h^{-1} \notin G_k$  (see Remark 2.5). Since  $ghg^{-1}h^{-1} \in H'$ , we see that  $H'$  is spread out and thus  $H' \neq \mathbf{1}$ .  $\square$

**Lemma 2.8** *Every non-trivial subnormal subgroup  $S$  of  $G$  is non-abelian and has trivial radical,  $R(S) = \mathbf{1}$ .*

**Proof.** Being subnormal and non-trivial,  $S$  is spread out (see Lemma 2.6) and hence non-abelian, by Lemma 2.7. Let  $S = S^{(0)} \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \dots$  be the derives series of  $S$ . Then  $S^{(n)}$  is a subnormal subgroup of  $G$ , for each  $n \in \mathbb{N}_0$ . We have  $S^{(0)} = S \neq \mathbf{1}$ . If  $S^{(n)} \neq \mathbf{1}$ , then  $S^{(n)}$  is spread out (by Lemma 2.6), whence  $S^{(n+1)} = (S^{(n)})' \neq \mathbf{1}$  by Lemma 2.7. Hence  $S^{(n)} \neq \mathbf{1}$  for all  $n \in \mathbb{N}_0$  and thus  $S$  is not soluble. If  $N$  is a non-trivial normal subgroup of  $S$ , then also  $N$  is subnormal in  $G$  and hence  $N$  is not soluble, by the preceding. Hence  $\mathbf{1}$  is the only soluble normal subgroup of  $S$  and hence  $R(S) = \mathbf{1}$ .  $\square$

### 3 Proof of the theorem

We now prove the theorem stated in the Introduction.

First note that  $G_0 := \mathbb{Q}_p$  is a  $p$ -adic Lie group with  $p^{-1}\mathbb{Z}_p$  as a central open subgroup. If  $n \in \mathbb{N}$  and we already know that  $G_{n-1}$  is a  $p$ -adic Lie group with  $p^{n-2}\mathbb{Z}_p$  as a central open subgroup, then  $G_n := G_{n-1} *_{p^{n-1}\mathbb{Z}_p} \mathbb{Q}_p$  has  $p^{n-1}\mathbb{Z}_p$  as a central subgroup. Now Proposition 18 in [1, Chapter III, §1, no. 9] shows that  $G_n$  admits a unique  $p$ -adic Lie group structure turning  $p^{n-1}\mathbb{Z}_p$  into an open Lie subgroup of  $G_n$ . Then  $G_{n-1}$  is an open Lie subgroup of  $G_n$  and thus  $G := \bigcup_{n \in \mathbb{N}_0} G_n$  admits a unique Lie group structure making each  $G_n$  an open Lie subgroup.

(a) If  $g \in G$ , then  $g \in G_n$  for some  $n \in \mathbb{N}_0$ . Since  $p^{n-1}\mathbb{Z}_p \subseteq Z(G_n)$ , the centre  $Z(G_n)$  is open in  $G_n$  and thus  $\text{Ad}(g) = \text{id}$  (as it does not matter for the calculation of  $\text{Ad}(g)$  if we consider  $g$  as an element of  $G$ , or as an element of the open subgroup  $G_n \subseteq G$ ).

(b) is a special case of Lemma 2.8.

(c) To construct a group homomorphism  $\phi: G \rightarrow \mathbb{Q}_p$ , we start with  $\phi_0 := \text{id}_{\mathbb{Q}_p}: G_0 = \mathbb{Q}_p \rightarrow \mathbb{Q}_p, z \mapsto z$ . If  $n \in \mathbb{N}$  and a homomorphism  $\phi_{n-1}: G_{n-1} \rightarrow \mathbb{Q}_p$  has already been constructed with  $\phi_{n-1}(z) = z$  for all  $z \in p^{n-2}\mathbb{Z}_p$ , then the universal property of the amalgamated product provides a unique group homomorphism

$$\phi_n: G_n = G_{n-1} *_{p^{n-1}\mathbb{Z}_p} \mathbb{Q}_p \rightarrow \mathbb{Q}_p \quad (8)$$

such that

$$\phi_n|_{G_{n-1}} = \phi_{n-1} \quad (9)$$

and  $\phi_n|_{\mathbb{Q}_p} = \text{id}_{\mathbb{Q}_p}: \mathbb{Q}_p \rightarrow \mathbb{Q}_p$  (considered as a map from the second factor in (8) to  $\mathbb{Q}_p$ ). By (9), we obtain a well-defined group homomorphism  $\phi: G \rightarrow \mathbb{Q}_p$  via  $\phi(x) := \phi_n(x)$  if  $x \in G_n$ . Then  $\phi|_{G_n} = \phi_n$  for each  $n$ . In particular, restricting  $\phi$  to  $G_0 = \mathbb{Q}_p$ , we obtain  $\phi|_{\mathbb{Q}_p} = \phi_0 = \text{id}_{\mathbb{Q}_p}$ . Since  $G_0$  is open in  $G$  and  $\text{id}_{\mathbb{Q}_p}$  an analytic map, we deduce that  $\phi$  is analytic and  $L(\phi) = T_e \text{id}_{\mathbb{Q}_p} = \text{id}_{\mathbb{Q}_p}$ . Thus  $\phi$  is étale and surjective, whence  $K := \ker \phi$  is discrete and  $G/K \cong \phi(G) = \mathbb{Q}_p$ . To get  $\psi$ , compose  $\phi$  with the embedding  $\mathbb{Q}_p \rightarrow \text{GL}_2(\mathbb{Q}_p), z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ .

(d) Since  $G/K$  is abelian in (c), we have  $G' \subseteq K$ . As  $K$  is discrete, discreteness of  $G'$  follows.  $\square$



## 4 Proof of the proposition

We now prove the proposition stated in the Introduction.

To reach a contradiction, suppose that an extraordinary  $p$ -adic Lie group  $H$  exists. If Problem 1 has a positive answer, then we can find a normal subgroup  $K \subseteq H$  such that  $H/K$  is linear and there exists an abelian, normal subgroup  $Z \subseteq K$  with  $K/Z$  discrete. Then  $Z$  is open in  $H$  or discrete. If  $Z$  is open, then  $Z$  is an abelian open subnormal subgroup of  $H$ ; but  $Z'$  must be non-discrete as  $H$  is extraordinary (contradiction). If  $Z$  is discrete, then also  $K$  is discrete. Because  $H/K$  is linear, we have  $L(R(H/K)) = \text{rad}(L(H/K))$  (see [4, Lemma 1.3]; cf. [2, Lemma 6.4]) and thus  $L(R(H/K)) = L(H/K)$ , since  $L(H/K)$  (like  $L(H)$ ) is abelian. Hence  $R(H/K)$  contains a soluble closed normal subgroup  $S$  of  $H/K$  (see [4, Lemma 1.2]). Let  $q: H \rightarrow H/K$  be the quotient morphism. Then  $T := q^{-1}(S)$  is an open subnormal subgroup of  $H$  and  $T^{(k)}$  is discrete for large  $k$  (because  $T^{(k)} \subseteq K$  for large  $k$ , in view of  $q(T^{(k)}) = S^{(k)}$ ). Choose  $k$  minimal with  $T^{(k)}$  discrete. Then  $N := T^{(k-1)}$  is an open subnormal subgroup of  $H$  such that  $N'$  is discrete, contradicting the hypothesis that  $H$  is extraordinary.  $\square$

## References

- [1] Bourbaki, N., “Lie Groups and Lie Algebras, Chapters 1–3,” Springer, Berlin, 1989.
- [2] Cluckers, R., Y. Cornulier, N. Louvet, R. Tessera and A. Valette, *The Howe-Moore property for real and  $p$ -adic Lie groups*, Math. Scand. **109** (2011), no. 2, 201–224.
- [3] Dixon, J.D., M.P.F. du Sautoy, A. Mann and D. Segal, “Analytic Pro- $p$  Groups,” Cambridge University Press, <sup>2</sup>1999
- [4] Glöckner, H., *Elementary  $p$ -adic Lie groups have finite constructible rank*, preprint, [arXiv:1402.4919](https://arxiv.org/abs/1402.4919).
- [5] Rotman, J.J., “An Introduction to the Theory of Groups,” Springer, New York, 1995.
- [6] Serre, J.-P., “Lie Algebras and Lie Groups,” Springer, Berlin, 1992.

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